

# $k$ -ORDERED HAMILTON CYCLES IN DIGRAPHS

DANIELA KÜHN, DERYK OSTHUS AND ANDREW YOUNG

**ABSTRACT.** Given a digraph  $D$ , let  $\delta^0(D) := \min\{\delta^+(D), \delta^-(D)\}$  be the minimum semi-degree of  $D$ .  $D$  is  $k$ -ordered Hamiltonian if for every sequence  $s_1, \dots, s_k$  of distinct vertices of  $D$  there is a directed Hamilton cycle which encounters  $s_1, \dots, s_k$  in this order. Our main result is that every digraph  $D$  of sufficiently large order  $n$  with  $\delta^0(D) \geq \lceil (n+k)/2 \rceil - 1$  is  $k$ -ordered Hamiltonian. The bound on the minimum semi-degree is best possible. An undirected version of this result was proved earlier by Kierstead, Sárközy and Selkow [10].

## 1. INTRODUCTION

The famous theorem of Dirac determines the smallest minimum degree of a graph which guarantees the existence of a Hamilton cycle. There are many subsequent results which investigate degree conditions that guarantee the existence of a Hamilton cycle with some additional properties. In particular, Chartrand (see [13]) introduced the notion of a Hamilton cycle which has to visit a given set of vertices in a prescribed order. More formally, we say that a graph  $G$  is  $k$ -ordered if for every sequence  $s_1, \dots, s_k$  of distinct vertices of  $G$  there is a cycle which encounters  $s_1, \dots, s_k$  in this order.  $G$  is  $k$ -ordered Hamiltonian if it contains a Hamilton cycle with this property. Kierstead, Sárközy and Selkow [10] showed that for all  $k \geq 2$  every graph on  $n \geq 11k - 3$  vertices of minimum degree at least  $\lceil n/2 \rceil + \lfloor k/2 \rfloor - 1$  is  $k$ -ordered Hamiltonian. This bound on the minimum degree is best possible and proved a conjecture of Ng and Schultz [13]. Several related problems have subsequently been considered: for instance, the case when  $k$  is large compared to  $n$  was investigated in [6] (but has not been completely settled yet). Ore-type conditions were investigated in [13, 6, 5]. For more results in this direction, see the survey by Gould [8].

It seems that digraphs provide an equally natural setting for such problems. Our main result is a version of the result in [10] for digraphs. The digraphs we consider do not have loops and we allow at most one edge in each direction between any pair of vertices. Given a digraph  $D$ , the *minimum semi-degree*  $\delta^0(D)$  of  $D$  is the minimum of the minimum outdegree  $\delta^+(D)$  of  $D$  and its minimum indegree  $\delta^-(D)$ .

**Theorem 1.** *For every  $k \geq 3$  there is an integer  $n_0 = n_0(k)$  such that every digraph  $D$  on  $n \geq n_0$  vertices with  $\delta^0(D) \geq \lceil (n+k)/2 \rceil - 1$  is  $k$ -ordered Hamiltonian.*

Our proof shows that one can take  $n_0 := Ck^9$  where  $C$  is a sufficiently large constant. Note that if  $n$  is even and  $k$  is odd the bound on the minimum semi-degree is slightly larger than in the undirected case. However, it is best possible in all cases. In fact, if the minimum semi-degree is smaller, it turns out that  $D$  need not even be  $k$ -ordered. This is easy to see if  $k$  is even: let  $D$  be the digraph which consists of a complete digraph  $A$  of order  $\lceil n/2 \rceil + k/2 - 1$  and a complete digraph  $B$  of order  $\lfloor n/2 \rfloor + k/2$  which has precisely  $k - 1$  vertices in common with  $A$ . Pick

vertices  $s_1, s_3, \dots, s_{k-1} \in A - B$  and  $s_2, s_4, \dots, s_k \in B - A$ . Then  $D$  has no cycle which encounters  $s_1, \dots, s_k$  in this order. A similar construction also works if both  $k$  and  $n$  are odd. The construction in the remaining case is a little more involved, see [11] for details. Note that every Hamiltonian digraph is 2-ordered Hamiltonian, so the case when  $k \leq 2$  in Theorem 1 is covered by the result of Ghouila-Houri [7] (Theorem 4 below) which implies that every digraph with minimum semi-degree at least  $n/2$  contains a Hamilton cycle.

Theorem 1 can be used to deduce a version for edges which have to be traversed in a prescribed order by the Hamilton cycle: we say that a digraph  $D$  is *k-arc ordered Hamiltonian* if, for every sequence  $e_1, \dots, e_k$  of independent edges,  $D$  contains a Hamilton cycle which encounters  $e_1, \dots, e_k$  in this order.  $D$  is *k-arc Hamiltonian* if it contains a Hamilton cycle which encounters these edges in any order.  $D$  is called *Hamiltonian k-linked* if  $|D| \geq 2k$  and if for every sequence  $x_1, \dots, x_k, y_1, \dots, y_k$  of distinct vertices there are disjoint paths  $P_1, \dots, P_k$  in  $D$  such that  $P_i$  joins  $x_i$  to  $y_i$  and such that together all the  $P_i$  cover all the vertices of  $D$ . Thus every digraph  $D$  which is Hamiltonian  $k$ -linked is also  $k$ -arc ordered Hamiltonian. Indeed, if  $x_1y_1, \dots, x_ky_k$  are the (directed) edges our Hamilton cycle has to encounter then disjoint paths linking  $y_{i-1}$  to  $x_i$  for all  $i = 1, \dots, k$  yield the required Hamilton cycle.

**Corollary 2.** *For all  $k \geq 3$  there is an integer  $n_0 = n_0(k)$  such that every digraph  $D$  on  $n \geq n_0$  vertices with  $\delta^0(D) \geq \lceil n/2 \rceil + k - 1$  is Hamiltonian  $k$ -linked and thus in particular  $k$ -arc ordered Hamiltonian.*

The examples in [11] show that in both parts of Corollary 2 the bound on the minimum semi-degree is best possible. In fact, if the minimum semi-degree is smaller then one cannot even guarantee the digraph to be  $k$ -arc ordered. A result of Bermond [3] (see also [2]) implies that if  $\delta^0(D) \geq \lceil (n+k)/2 \rceil$  then  $D$  is  $k$ -arc Hamiltonian. It easily follows that if  $\delta^0(D) \geq \lceil (n+1)/2 \rceil$ , then  $D$  is Hamiltonian 1-linked, i.e. Hamiltonian connected (see [2]). This covers the case  $k = 1$  of Corollary 2. As observed in [1, Thm 9.2.10], if  $\delta^0(D) \geq \lceil n/2 \rceil + 1$ , then  $D$  is Hamiltonian 2-linked, which covers the case  $k = 2$  of Corollary 2.

Corollary 2 can easily be deduced from Theorem 1 as follows: let  $x_1, \dots, x_k$  and  $y_1, \dots, y_k$  be distinct vertices where we aim to link  $x_i$  to  $y_i$  for all  $i$ . Let  $D'$  be the digraph obtained from  $D$  by contracting  $x_i$  and  $y_{i-1}$  into a new vertex  $s_i$  whose outneighbourhood is that of  $x_i$  and whose inneighbourhood is that of  $y_{i-1}$ . More precisely, let  $A := \{x_1, \dots, x_k, y_1, \dots, y_k\}$ . Then  $D'$  is the digraph obtained from  $D - A$  by adding new vertices  $s_1, \dots, s_k$  and defining the edges incident to these new vertices as follows. The outneighbours of  $s_i$  are the outneighbours of  $x_i$  in  $V(D) \setminus A$  as well as all the  $s_j$  for all those  $j \neq i-1$  for which  $y_j$  is an outneighbour of  $x_i$  in  $D$  (where  $y_0 := y_k$ ). Similarly, inneighbours of  $s_i$  are the inneighbours of  $y_{i-1}$  in  $V(D) \setminus A$  as well as all the  $s_j$  for all those  $j \neq i$  for which  $x_j$  is an inneighbour of  $y_{i-1}$  in  $D$ . It is easy to check that  $\delta^0(D') \geq \lceil (|D'| + k)/2 \rceil - 1$  and that a Hamilton cycle in  $D'$  which encounters  $s_1, \dots, s_k$  in this order corresponds to a spanning set of disjoint paths from  $x_i$  to  $y_i$ .

A result of Chen et al. [4, Theorem 10] implies that the smallest minimum degree which guarantees an undirected graph to be  $k$ -arc ordered Hamiltonian is  $\lfloor n/2 \rfloor + k - 1$ . (A graph is  $k$ -arc ordered Hamiltonian if for any sequence of  $k$

independent oriented edges there exists a Hamilton cycle which encounters these edges in the given order and orientation.) The smallest minimum degree which forces a graph to be  $k$ -linked was determined by Kawarabayashi, Kostochka and Yu [9]. It is not clear whether the minimum degree for Hamiltonian  $k$ -linkedness is the same.

The main tool in our proof of Theorem 1 is a recent result by the first authors (Theorem 3 below), which shows that the degree condition in Theorem 1 at least guarantees a  $k$ -ordered cycle (but not necessarily a Hamiltonian one). The strategy of the proof of Theorem 1 is to consider such a cycle of maximal length and to show that it must be Hamiltonian. The same strategy was already applied in the proof of the undirected case in [10]. However, both parts of the strategy are more difficult in the digraph case: the existence of a  $k$ -ordered directed cycle (i.e. Theorem 3) already confirms a conjecture of Manoussakis [12] for large  $n$ . The Hamiltonicity of a  $k$ -ordered cycle of maximal length is easier to show in the undirected case as one can consider ‘local transformations’ of a given  $k$ -ordered cycle which reverse the orientation of certain segments of the cycle. This means that apart from some basic observations like Lemma 8 below our proof is quite different from that in [10].

**Theorem 3.** [11] *Let  $k$  and  $n$  be integers such that  $k \geq 2$  and  $n \geq 200k^3$ . Then every digraph  $D$  on  $n$  vertices with  $\delta^0(D) \geq \lceil (n+k)/2 \rceil - 1$  is  $k$ -ordered.*

## 2. NOTATION AND TOOLS

Given a digraph  $D$ , we write  $V(D)$  for its vertex set,  $E(D)$  for its edge set and  $|D| := |V(D)|$  for its order. We write  $xy$  for the edge directed from  $x$  to  $y$ . More generally, if  $A$  and  $B$  are disjoint sets of vertices of  $D$  then an  $A$ - $B$  edge is an edge of the form  $ab$  where  $a \in A$  and  $b \in B$ . A digraph is *complete* if every pair of distinct vertices is joined by edges in both directions.

Given disjoint subdigraphs  $D_1$  and  $D_2$  of a digraph  $D$  such that  $D_1 \cup D_2$  is spanning and a set  $A \subseteq V(D_1)$ , we write  $N_{D_i}^+(A)$  for the set of all those vertices  $x \in V(D_i) \setminus A$  which in the digraph  $D$  receive an edge from some vertex in  $A$ .  $N_{D_i}^-(A)$  is defined similarly. If  $A$  consists of a single vertex  $x$ , we just write  $N_{D_i}^+(x)$  etc. and put  $d_{D_i}^+(x) := |N_{D_i}^+(x)|$  and  $d_{D_i}^-(x) := |N_{D_i}^-(x)|$ . So in particular,  $N_D^+(x)$  is the outneighbourhood of  $x$  in  $D$  and  $d_D^+(x)$  is its outdegree. Also, note that  $N_{D_1}^+(x)$  is the outneighbourhood of  $x$  in the subdigraph  $D[V(D_1)]$  of  $D$  induced by  $V(D_1)$  and not its outneighbourhood in  $D_1$  (where  $x \in D_1$ ). We let  $N_D(x) := N_D^+(x) \cup N_D^-(x)$ .

If we refer to paths and cycles in digraphs then we always mean that they are directed without mentioning this explicitly. The *length* of a path is the number of its edges. Given two vertices  $x, y \in D$ , an  $x$ - $y$  path is a path which is directed from  $x$  to  $y$ . Given two vertices  $x$  and  $y$  on a directed cycle  $C$ , we write  $xCy$  for the subpath of  $C$  from  $x$  to  $y$ . Similarly, given two vertices  $x$  and  $y$  on a directed path  $P$  such that  $x$  precedes  $y$ , we write  $xPy$  for the subpath of  $P$  from  $x$  to  $y$ .

A digraph  $D$  is *strongly connected* if for every ordered pair  $x, y$  of vertices of  $D$  there exists an  $x$ - $y$  path.  $D$  is *Hamiltonian connected* if for every ordered pair  $x, y$  of vertices of  $D$  there exists a Hamilton path from  $x$  to  $y$ . (So Hamiltonian connectedness is the same as Hamiltonian 1-linkedness.)

We will often use the following result of Ghouila-Houri [7] which gives a sufficient condition for the existence of a Hamilton cycle in a digraph. In particular, it implies a version of Theorem 1 for  $k \leq 2$  as any Hamiltonian digraph is 2-ordered Hamiltonian.

**Theorem 4.** *Suppose that  $D$  is a strongly connected digraph such that  $d_D^+(x) + d_D^-(x) \geq |D|$  for every vertex  $x \in D$ . Then  $D$  is Hamiltonian.*

The next result of Overbeck-Larisch [14] provides a sufficient condition for a digraph to be Hamiltonian connected.

**Theorem 5.** *Suppose that  $D$  is a digraph such that  $d_D^+(x) + d_D^-(y) \geq |D| + 1$  whenever  $xy$  is not an edge. Then  $D$  is Hamiltonian connected.*

### 3. PRELIMINARY RESULTS

Let  $D$  be a digraph satisfying the conditions of Theorem 1. Let  $S = (s_1, \dots, s_k)$  by any sequence of  $k \geq 3$  vertices of  $D$ . We will often view  $S$  as a set. An  $S$ -cycle in  $D$  is a cycle which encounters  $s_1, \dots, s_k$  in this order. So we have to show that  $D$  has a Hamiltonian  $S$ -cycle. Theorem 3 implies the existence of an  $S$ -cycle in  $D$ . Let  $C$  be a longest such cycle and suppose that  $C$  is not Hamiltonian. Let  $H$  be the subdigraph of  $D$  induced by all the vertices outside  $C$ . Our aim is to find a longer  $S$ -cycle by modifying  $C$  (yielding a contradiction). The purpose of this section is to collect the properties of  $C$  and  $H$  that we need in our proof of Theorem 1.

We let  $F$  be the set of all those vertices on  $C$  which receive an edge from some vertex in  $H$  and we let  $T$  be the set of all those vertices on  $C$  which send an edge to some vertex in  $H$ . Given  $i \in \mathbb{N}$ , we write  $F_i$  for the set of all those vertices on  $C$  which receive an edge from at least  $i$  vertices in  $H$ . Thus  $F_1 = F$ .  $T_i$  is defined similarly. Given a vertex  $x$  on  $C$ , we will denote its successor on  $C$  by  $x^+$  and its predecessor by  $x^-$ .

**Lemma 6.**  *$H$  is Hamiltonian connected and  $d_H^-(x) + d_H^+(y) \geq |H| + k - 2$  for all vertices  $x, y \in H$ . Moreover any digraph obtained from  $H$  by deleting at most 2 vertices is strongly connected and  $k \leq |H| \leq \lfloor \frac{n-k}{2} \rfloor$ .*

**Proof.** We first show that any two (not necessarily distinct) vertices  $x, y \in H$  for which  $H$  contains an  $x$ - $y$  path,  $P$  say, satisfy the degree condition in the lemma. To see this, note that no vertex in  $N_C^-(x)$  is a predecessor of some vertex in  $N_C^+(y)$ . Indeed, if  $v \in N_C^-(x)$  and  $v^+ \in N_C^+(y)$  then by replacing the edge  $vv^+$  with the path  $vxPyv^+$  we obtain a longer  $S$ -cycle, a contradiction. But this means that  $d_C^-(x) + d_C^+(y) \leq |C|$  and thus

$$(1) \quad d_H^-(x) + d_H^+(y) \geq 2 \left( \left\lceil \frac{n+k}{2} \right\rceil - 1 \right) - |C| \geq |H| + k - 2,$$

as required. However, as  $k \geq 3$  this degree condition means that  $N_H^-(x) \cap N_H^+(y) \neq \emptyset$  and so  $H$  contains an  $y$ - $x$  path of length 2. Thus whenever  $H$  contains an  $x$ - $y$  path it also contains a  $y$ - $x$  path.

Now let  $x$  and  $z$  be any two vertices of  $H$ . What we have shown above applied with  $y := x$  implies that  $d_H^-(x) + d_H^+(x) \geq |H| + 1$  and thus  $|N_H(x)| \geq (|H| + 1)/2$ . Note that by the above  $x$  is joined to every vertex in  $N_H(x)$  with paths in both directions. Similarly,  $|N_H(z)| \geq (|H| + 1)/2$  and  $z$  is joined to every vertex in  $N_H(z)$

with paths in both directions. As  $|N_H(x) \cap N_H(z)| > 0$  this means that  $x$  is joined to  $z$  with paths in both directions, i.e.  $H$  is strongly connected. Together with (1) this in turn implies that  $d_H^-(x) + d_H^+(z) \geq |H| + k - 2 \geq |H| + 1$  for all vertices  $x, z \in H$ . In particular,  $H$  is Hamiltonian connected by Theorem 5.

To show that any digraph  $H'$  obtained from  $H$  by deleting at most 2 vertices is strongly connected note that  $d_{H'}^-(x) + d_{H'}^+(y) \geq |H'| - 1$  for every  $x, y \in H'$ . Thus if  $x \neq y$  then either  $yx$  is an edge or  $H'$  contains an  $y$ - $x$  path of length 2.

It now remains to prove the bounds on  $|H|$ . Consider any vertex  $x \in H$ . Then  $2(|H| - 1) \geq d_H^-(x) + d_H^+(x) \geq |H| + k - 2$  and so  $|H| \geq k$ . For the upper bound, note that no vertex in  $T$  has a successor in  $F$ . Indeed, if  $v$  is such a vertex in  $T$  and  $v^+$  is its successor then we could replace  $vv^+$  with a path through  $H$  to obtain a longer  $S$ -cycle, a contradiction. But this means that some vertex of  $C$  must have all its inneighbours on  $C$  or all its outneighbours on  $C$ . Thus  $|C| \geq \lceil (n + k)/2 \rceil$  and so  $|H| \leq \lfloor (n - k)/2 \rfloor$ .  $\square$

Recall that the proof of Lemma 6 implies the following.

**Corollary 7.** *No vertex on  $C$  which lies in  $T$  has a successor in  $F$ .*

The next result deals with the case when the vertices  $x_1 \in T$  and  $x_2 \in F$  are further apart.

**Lemma 8.** *Suppose that  $x_1, x_2 \in C$  are distinct and the interior of  $x_1Cx_2$  does not contain a vertex from  $S$ . Then there are no distinct vertices  $y_1, y_2 \in H$  such that  $x_1y_1, y_2x_2 \in E(D)$ .*

**Proof.** Suppose that such  $y_1, y_2$  do exist. Furthermore, we may assume that  $x_1$  and  $x_2$  are chosen such that they satisfy all these properties and subject to this  $|x_1Cx_2|$  is minimum. Let  $Q$  denote the set of all vertices in the interior of  $x_1Cx_2$ . Then our choice of  $x_1$  and  $x_2$  implies that  $N_C^-(y_1) \cap Q = \emptyset$  and  $N_C^+(y_2) \cap Q = \emptyset$ . Moreover, by Corollary 7 no vertex in  $N_C^-(y_1)$  is a predecessor of some vertex in  $N_C^+(y_2)$ . Thus  $d_C^-(y_1) + d_C^+(y_2) \leq |C| - |Q| + 1$  and so

$$n + k - 2 \leq d_D^-(y_1) + d_D^+(y_2) \leq |C| - |Q| + 1 + 2(|H| - 1) = n - |Q| + |H| - 1.$$

This implies that  $|H| > |Q|$  and thus replacing the interior of  $x_1Cx_2$  with a Hamilton path from  $y_1$  to  $y_2$  through  $H$  (which exists by Lemma 6) yields a longer  $S$ -cycle, a contradiction.  $\square$

The next two results will be used in the proof of Lemma 11.

**Lemma 9.** *Let  $G$  be a digraph such that  $d_G^+(x) + d_G^-(x) \geq |G| + 3$  for every vertex  $x \in G$  and  $d_G^+(x) + d_G^-(y) \geq |G| + 1$  for every pair of vertices  $x, y \in G$ . Let  $z_1$  and  $z_2$  be distinct vertices of  $G$  such that  $z_1z_2 \notin E(G)$ . Then there exists a vertex  $a \in N_G^+(z_1) \cap N_G^-(z_2)$  such that  $G - \{z_1, z_2, a\}$  is strongly connected.*

**Proof.** First note that  $|N_G^+(z_1) \cap N_G^-(z_2)| \geq 3$  since  $z_1z_2 \notin E(G)$ . Pick  $a_1, a_2, a_3 \in N_G^+(z_1) \cap N_G^-(z_2)$ . We will show that one of these  $a_i$  can play the role of  $a$ . Let  $G^* := G - \{z_1, z_2\}$ . Note that  $d_{G^*}^+(x) + d_{G^*}^-(x) \geq |G^*| + 1$  for every vertex  $x \in G^*$  and  $d_{G^*}^+(x) + d_{G^*}^-(y) \geq |G^*| - 1$  for every pair of vertices  $x, y \in G^*$ . In particular, the latter condition implies that  $G^*$  is strongly connected. Thus  $G^*$  has a Hamilton cycle  $C$  by Theorem 4. Let  $a_1^+$  denote the successor of  $a_1$  on  $C$  and let  $a_1^-$  be its

predecessor. Put  $N^+ := N_{G^*}^+(a_1^-) \setminus \{a_1\}$  and  $N^- := N_{G^*}^-(a_1^+) \setminus \{a_1\}$ . Note that  $|N^+|, |N^-| \geq 1$  since  $d_{G^*}^+(a_1^-) + d_{G^*}^-(a_1^-) \geq |G^*| + 1$  and  $d_{G^*}^+(a_1^+) + d_{G^*}^-(a_1^+) \geq |G^*| + 1$ . Similarly  $|N^+| + |N^-| \geq |G^*| - 3$ . Clearly, if  $a_1^- a_1^+$  is an edge or  $N^+ \cap N^- \neq \emptyset$ , then  $G^* - a_1$  is strongly connected and so we can take  $a$  to be  $a_1$ . So we may assume that neither of these is the case. But then  $N^+ \cup N^- = V(G^*) \setminus \{a_1, a_1^+, a_1^-\}$ . Let  $v \in N^+$  be such that  $|vCa_1^-|$  is maximal. Similarly, let  $w \in N^-$  be such that  $|a_1^+Cw|$  is maximal. Note that if  $w \in vCa_1^-$  then  $G^* - a_1$  is strongly connected. So we may assume that this is not the case. But then  $v$  must be the successor of  $w$  on  $C$ ,  $N^+$  must consist of precisely the vertices in  $V(vCa_1^-) \setminus \{a_1^-\}$  and  $N^-$  must consist of precisely the vertices in  $V(a_1^+Cw) \setminus \{a_1^+\}$ .

Let  $A^+ := N^+ \cup \{a_1^-\}$  and  $A^- := N^- \cup \{a_1^+\}$ . We may assume that  $G$  does not contain an  $A^+ - A^-$  edge as otherwise  $G^* - a_1$  is strongly connected. We will now show  $G^*[A^+]$  is complete and that  $a_1$  receives an edge from every vertex in  $A^+$ . So consider any vertex  $x \in A^+$ . Then  $d_{G^*}^+(x) + d_{G^*}^-(a_1^+) \geq |G^*| - 1$ . Together with the fact that there is no  $A^+ - A^-$  edge this shows that  $N_{G^*}^+(x) = (A^+ \cup \{a_1\}) \setminus \{x\}$ . Thus  $G^*[A^+]$  is complete and  $a_1$  receives an edge from every vertex in  $A^+$ . Similarly one can show that  $G^*[A^-]$  is complete and that  $a_1$  sends an edge to every vertex in  $A^-$ .

Now consider  $a_2$  and  $a_3$ . If for example  $a_2 \neq v, w$  then  $G^* - a_2$  is strongly connected and so we can take  $a$  to be  $a_2$ . As one can argue similarly for  $a_3$ , we may assume that  $v = a_2$  and  $w = a_3$ . If  $a_1^+ a_1^-$  is an edge or  $a_1 \in N_{G^*}^+(a_1^+) \cap N_{G^*}^-(a_1^-)$  then  $G^* - a_2$  is strongly connected. (Here we used that  $a_1^- \neq v = a_2$  since  $|N^+| \geq 1$ .) If this is not the case, then  $d_{G^*}^+(a_1^+) + d_{G^*}^-(a_1^-) \geq |G^*| - 1$  implies the existence of some vertex  $x \in N_{G^*}^+(a_1^+) \cap N_{G^*}^-(a_1^-)$  with  $x \neq a_1$ . If  $x \in A^+$  then  $a_1^+ x$  is an  $A^- - A^+$  edge avoiding  $w = a_3$  and so  $G^* - a_3$  is strongly connected. (Here we used that  $a_1^+ \neq w = a_3$  since  $|N^-| \geq 1$ .) Similarly, if  $x \in A^-$  then  $G^* - a_2$  is strongly connected. Altogether, this shows that we can take  $a$  to be  $a_1, a_2$  or  $a_3$ .  $\square$

**Lemma 10.** *Suppose that  $H$  contains a vertex  $v$  with  $d_H^-(v) + d_H^+(v) \leq |H| + k - 1$ . Suppose that  $x_1, x_2 \in T$  and  $y_1, y_2 \in F$  are distinct vertices on  $C$ . Then  $x_1 v, v y_1 \in E(D)$  or  $x_2 v, v y_2 \in E(D)$  (or both).*

**Proof.** Let  $F_v$  denote the set of all those vertices on  $C$  which receive an edge from  $v$ . Let  $T_v^+$  denote the set of all those vertices on  $C$  whose predecessor sends an edge to  $v$ . Corollary 7 implies that  $T_v^+ \cap F_v = \emptyset$ . Since

$$d_C^-(v) + d_C^+(v) \geq 2 \left( \left\lceil \frac{n+k}{2} \right\rceil - 1 \right) - (|H| + k - 1) \geq |C| - 1$$

this shows that at most one vertex on  $C$  lies outside  $T_v^+ \cup F_v$ . Let  $z$  be the vertex in  $V(C) \setminus (T_v^+ \cup F_v)$  (if it exists).

Suppose first that  $z \notin F$  (this also covers the case when  $z$  does not exist). Then  $z \neq y_1, y_2$ . Also either  $z \neq x_1^+$  or  $z \neq x_2^+$ . So let us assume that  $z \neq x_1^+$  (the case when  $z \neq x_2^+$  is similar). We will show that  $x_1 v, v y_1 \in E(D)$ . So suppose first that  $x_1 v \notin E(D)$ . Then  $x_1^+ \notin T_v^+$  and thus  $x_1^+ \in F_v$ , a contradiction to Corollary 7. Similarly, if  $v y_1 \notin E(D)$  then  $y_1 \notin F_v$  and thus  $y_1 \in T_v^+$ , i.e. the predecessor of  $y_1$  lies in  $T$ , contradicting Corollary 7.

So suppose next that  $z \in F$  and thus, by Corollary 7, the predecessor of  $z$  does not lie in  $T$ . This in turn implies that  $z \neq x_1^+, x_2^+$ . Moreover either  $z \neq y_1$  or  $z \neq y_2$ . So let us assume that  $z \neq y_1$ . Similarly as before one can show that  $x_1v, vy_1 \in E(D)$ .  $\square$

In our proof of Theorem 1 we will frequently need two disjoint paths through  $H$  joining two given disjoint pairs of vertices on  $C$  in order to modify  $C$  into a longer  $S$ -cycle. The following lemma implies the existence of such paths provided that the pairs consist of vertices having sufficiently many neighbours in  $H$  (see also Corollary 12).

**Lemma 11.** *Suppose that  $X_1, X_2 \subseteq T$  and  $Y_1, Y_2 \subseteq F$  are disjoint subsets of  $V(C)$  such that  $|N_H^+(X_1)|, |N_H^+(X_2)| \geq 3$  and  $|N_H^-(Y_1)|, |N_H^-(Y_2)| \geq 3$ . Then there are disjoint  $X_i$ - $Y_i$  paths  $P_i$  of length at least 2 and such that all inner vertices of  $P_1$  and  $P_2$  lie in  $H$ . Moreover, if  $|H| \geq 15$  and if we even have that  $|N_H^+(X_1)|, |N_H^+(X_2)| \geq 8$  and  $|N_H^-(Y_1)|, |N_H^-(Y_2)| \geq 8$  then we can find such paths which additionally satisfy  $|P_1 \cup P_2| \geq |H|/6$ .*

**Proof.** By disregarding some neighbours if necessary we may assume that  $|N_H^+(X_1)| = |N_H^+(X_2)| = |N_H^-(Y_1)| = |N_H^-(Y_2)|$ . Our first aim is to show that for some  $i \in \{1, 2\}$  there is an  $X_i$ - $Y_i$  path  $P_i$  which satisfies the following properties:

- (i) The graph  $H' := H - V(P_i)$  has a Hamilton cycle  $C'$ .
- (ii) All  $x, y \in H'$  satisfy  $d_{H'}^+(x) + d_{H'}^-(y) \geq |H'| - 2$ .
- (iii)  $3 \leq |P_i| \leq 5$ , i.e.  $P_i$  contains at least 1 and at most 3 vertices from  $H$ .
- (iv) If  $i = 1$  then  $|N_H^+(X_2) \cap V(P_1)| \leq 2$  and  $|N_H^-(Y_2) \cap V(P_1)| \leq 2$ . If  $i = 2$  then  $|N_H^+(X_1) \cap V(P_2)| \leq 2$  and  $|N_H^-(Y_1) \cap V(P_2)| \leq 2$ .

If we have found such an  $i$ , say  $i = 1$ , then our aim is to use the Hamilton cycle  $C'$  in order to find  $P_2$ . To prove the existence of such an  $i$ , recall that Lemma 6 implies  $d_H^-(x) + d_H^+(y) \geq |H| + k - 2 \geq |H| + 1$  for every pair of vertices  $x, y \in H$ . Thus condition (ii) will hold automatically if (iii) holds.

Now suppose first that there exists a vertex  $z_1 \in N_H^+(X_1) \cap N_H^-(Y_1)$ . Take  $i = 1$  and take  $P_1$  to be any  $X_1$ - $Y_1$  path whose interior consists precisely of  $z_1$ . Then  $d_{H'}^-(x) + d_{H'}^+(x) \geq |H'|$  for every  $x \in H'$ . As  $H'$  is strongly connected by Lemma 6 we can apply Theorem 4 to find a Hamilton cycle  $C'$  of  $H'$ . (If  $|H'| = 2$  then  $C'$  will consist of just a double edge.) In the case when  $N_H^+(X_2) \cap N_H^-(Y_2) \neq \emptyset$  we proceed similarly.

Now suppose that  $N_H^+(X_1) \cap N_H^-(Y_1) = \emptyset$  and  $N_H^+(X_2) \cap N_H^-(Y_2) = \emptyset$ . Then Lemma 10 implies that  $d_H^-(x) + d_H^+(x) \geq |H| + k \geq |H| + 3$  for every  $x \in H$ . If there is an  $N_H^+(X_1)$ - $N_H^-(Y_1)$  edge  $z_1z_2$  take  $i := 1$  and take  $P_1$  to be any  $X_1$ - $Y_1$  path whose interior consists of this edge. Then  $d_{H'}^-(x) + d_{H'}^+(x) \geq |H| - 1 = |H'| + 1$  for every  $x \in H'$  and so again, as  $H'$  is strongly connected by Lemma 6, we can apply Theorem 4 to find a Hamilton cycle  $C'$  of  $H'$ . In the case when there is an  $N_H^+(X_2)$ - $N_H^-(Y_2)$  edge we proceed similarly.

Thus we may assume that  $N_H^+(X_i) \cap N_H^-(Y_i) = \emptyset$  and that there is no  $N_H^+(X_i)$ - $N_H^-(Y_i)$  edge (for  $i = 1, 2$ ). Pick any vertex  $z_1 \in N_H^+(X_1)$  and let  $z_2 \in N_H^-(Y_1)$  be a vertex such that  $|N_H^+(X_2) \cap \{z_1, z_2\}| \leq 1$  and  $|N_H^-(Y_2) \cap \{z_1, z_2\}| \leq 1$ . (The fact that we can choose such a  $z_2$  follows from  $N_H^+(X_i) \cap N_H^-(Y_i) = \emptyset$  and our assumption that

the sizes of the  $N_H^+(X_i)$  and the  $N_H^-(Y_i)$  are equal.) Apply Lemma 9 with  $G := H$  to find a vertex  $z_3 \in N_H^+(z_1) \cap N_H^-(z_2)$  such that  $H - \{z_1, z_2, z_3\}$  is strongly connected. Take  $i := 1$  and  $P_1$  to be any  $X_1$ - $Y_1$  path whose interior consists of  $z_1 z_3 z_2$ . Then  $d_{H'}^-(x) + d_{H'}^+(x) \geq |H| - 3 = |H'|$  for every  $x \in H'$  and so again  $H'$  contains a Hamilton cycle  $C'$  by Theorem 4. Our choice of  $z_1$  and  $z_2$  implies that (iv) holds.

Altogether, this shows that in each case for some  $i$  there exists a path  $P_i$  satisfying (i)–(iv). We may assume that  $i = 1$ . As mentioned before, our aim now is to use the Hamilton cycle  $C'$  of  $H'$  in order to find an  $X_2$ - $Y_2$  path  $P_2$  through  $H'$ . In the case when  $|N_H^+(X_2)|, |N_H^-(Y_2)| \geq 3$  this is trivial since by (iv) both  $N_H^+(X_2)$  and  $N_H^-(Y_2)$  meet  $H'$  in at least one vertex.

So suppose now that  $|H| \geq 15$  and  $|N_H^+(X_2)|, |N_H^-(Y_2)| \geq 8$  and thus we wish to find a long  $X_2$ - $Y_2$  path. To do this, let  $N^+ := N_H^+(X_2) \cap V(H')$  and  $N^- := N_H^-(Y_2) \cap V(H')$ . Thus  $|N^+|, |N^-| \geq 6$  by (iv). Choose  $a_1 \in N^+$  and  $b_1 \in N^-$  to be distinct such that  $|a_1 C' b_1|$  is maximum. If  $|a_1 C' b_1| \geq |H'|/6$  then we can take  $P_2$  to be any  $X_2$ - $Y_2$  path whose interior consists of  $a_1 C' b_1$ . So we may assume that  $|a_1 C' b_1| \leq |H'|/6$ .

Note that the choice of  $a_1$  and  $b_1$  implies that  $N^+, N^- \subseteq V(a_1 C' b_1)$ . Moreover, all the vertices in  $N^+$  must precede the vertices in  $N^-$  on  $a_1 C' b_1$ . (Indeed, if e.g.  $a \in N^+$  and  $b \in N^-$  are distinct vertices such that  $b$  precedes  $a$ , i.e.  $a$  lies on  $b C' b_1$  then  $|a C' b| \geq |H'| - |a_1 C' b_1| \geq |H'|/2$ , contradicting the choice of  $a_1$  and  $b_1$ .) Thus  $|N^+ \cap N^-| \leq 1$  and there are vertices  $a_2, \dots, a_5 \in N^+$  and  $b_2, \dots, b_5 \in N^-$  such that  $a_1, \dots, a_5, b_5, \dots, b_1$  are distinct and appear on  $C'$  in this order. We now distinguish several cases.

**Case 1.** *There are  $i, j \leq 4$  such that  $a_i b_j$  is an edge.*

Note that  $d_{H'}^+(a_5) \geq |H'|/2 - 1$  or  $d_{H'}^-(b_5) \geq |H'|/2 - 1$  by (ii). Suppose that the former holds (the other case is similar). As  $|a_1 C' b_1| \leq |H'|/6$  this means that  $a_5$  has at least  $|H'|/3 - 1$  outneighbours in the interior of  $b_1 C' a_1$  and so we can find such an outneighbour  $v$  with  $|v C' a_1| \geq |H'|/3$ . But then we can take  $P_2$  to be any  $X_2$ - $Y_2$  path whose interior consists of  $a_5 v C' a_i b_j$  (Figure 1).

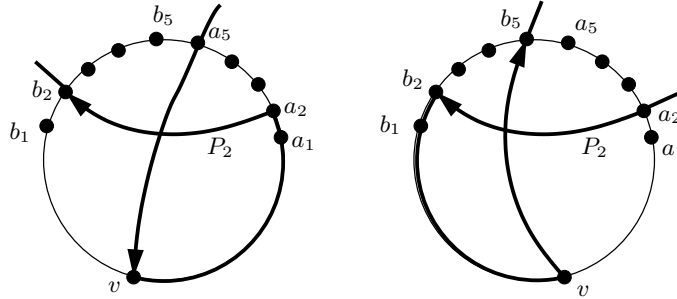


FIGURE 1. The path  $P_2$  in Case 1. The left figure is for the subcase when  $d_{H'}^+(a_5) \geq |H'|/2 - 1$  and the right figure is for the subcase when  $d_{H'}^-(b_5) \geq |H'|/2 - 1$ .

**Case 2.** *For all  $i, j \leq 4$   $a_i b_j$  is not an edge.*

**Case 2.1.** *There exists some vertex  $u \in N_{H'}^+(a_1) \cap N_{H'}^-(b_3)$ .*



Note that  $u \neq a_2, b_4$  since by our assumption neither  $a_2b_3$  nor  $a_1b_4$  is an edge. As before, either  $d_{H'}^+(a_2) \geq |H'|/2 - 1$  or  $d_{H'}^-(b_4) \geq |H'|/2 - 1$ . Suppose that the former holds (the other case is similar).

If  $u$  lies in the interior of  $a_1C'b_3$ , let  $v$  be an outneighbour of  $a_2$  in the interior of  $b_3C'a_1$  with  $|vC'a_1| \geq |H'|/3$ . Then we can take  $P_2$  to be any  $X_2$ - $Y_2$  path whose interior consists of  $a_2vC'a_1ub_3$ .

So we may assume that  $u$  lies in the interior of  $b_3C'a_1$ . But then either the interior of  $b_3C'u$  contains at least  $|H'|/6 - 1$  outneighbours of  $a_2$  or the interior of  $uC'a_1$  contains at least  $|H'|/6 - 1$  outneighbours of  $a_2$ . If the former holds let  $v$  be any outneighbour of  $a_2$  in the interior of  $b_3C'u$  such that  $|vC'u| \geq |H'|/6$  and take  $P_2$  to be any  $X_2$ - $Y_2$  path whose interior consists of  $a_2vC'ub_3$  (see Figure 2). If the latter holds let  $v$  be any outneighbour of  $a_2$  in the interior of  $uC'a_1$  such that  $|vC'a_1| \geq |H'|/6$  and take  $P_2$  to be any  $X_2$ - $Y_2$  path whose interior consists of  $a_2vC'a_1ub_3$ .

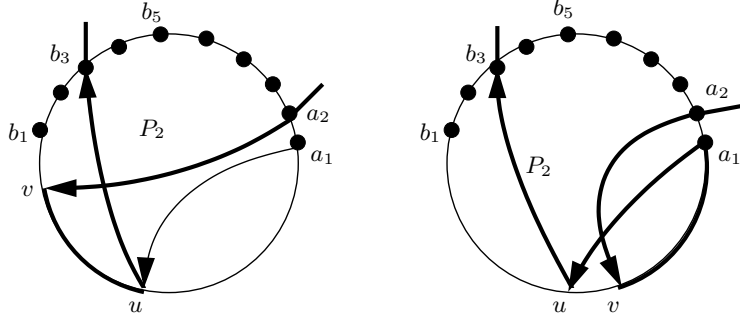


FIGURE 2. The path  $P_2$  in Case 2.1 if  $u$  lies in the interior of  $b_3C'a_1$ . The left figure is for the subcase when the interior of  $b_3C'u$  contains at least  $|H'|/6 - 1$  outneighbours of  $a_2$ . The right figure is for the subcase when the interior of  $uC'a_1$  contains at least  $|H'|/6 - 1$  outneighbours of  $a_2$ .

**Case 2.2.** *There exists some vertex  $u \in N_{H'}^+(a_3) \cap N_{H'}^-(b_1)$ .*

This case is similar to Case 2.1 and we omit the details.

**Case 2.3.** *Both  $N_{H'}^+(a_1) \cap N_{H'}^-(b_3)$  and  $N_{H'}^+(a_3) \cap N_{H'}^-(b_1)$  are empty.*

Together with (ii) and our assumption that  $a_1b_3$  is not an edge this implies that  $N_{H'}^+(a_1) \cup N_{H'}^-(b_3) = V(H') \setminus \{a_1, b_3\}$ . Since  $a_3b_3$  is not an edge this means that  $a_1a_3$  is an edge. Similarly it follows that  $b_3b_1$  is an edge. But as before either  $d_{H'}^+(a_2) \geq |H'|/2 - 1$  or  $d_{H'}^-(b_2) \geq |H'|/2 - 1$ . Suppose that the former holds (the other case is similar). Then we can find an outneighbour  $v$  of  $a_2$  in the interior of  $b_1C'a_1$  with  $|vC'a_1| \geq |H'|/3$ . But then we can take  $P_2$  to be any  $X_2$ - $Y_2$  path whose interior consists of  $a_2vC'a_1a_3C'b_1$ .  $\square$

Lemma 11 immediately implies the following corollary, which is sometimes more convenient to apply.

**Corollary 12.** *Suppose that  $x_1, x_2 \subseteq T_3$  and  $y_1, y_2 \subseteq F_3$  are distinct vertices on  $C$ . Then  $D$  contains disjoint  $x_i$ - $y_i$  paths  $P_i$  of length at least 2 such that all*

inner vertices of  $P_1$  and  $P_2$  lie in  $H$ . Moreover, if  $|H| \geq 15$  and if we even have that  $x_1, x_2 \subseteq T_8$  and  $y_1, y_2 \subseteq F_8$  then we can find such paths which additionally satisfy  $|P_1 \cup P_2| \geq |H|/6$ .

The last of our preliminary results gives a lower bound on the sizes of  $T_3$  and  $F_3$ .

**Lemma 13.** *We have that  $|T|, |F| \geq (n+k)/2 - |H|$ . Moreover,  $|T_3|, |F_3| \geq (n-k)/2 - |H|$  and  $|T_3 \cup F_3| \geq |C| - |H| - 2k$ .*

**Proof.** To see the bound on  $|T|$ , note that  $d_C^-(x) \geq \delta^0(D) - (|H| - 1) \geq (n+k)/2 - |H|$  for every vertex  $x \in H$  and so  $|T| \geq (n+k)/2 - |H|$ . The proof for  $|F|$  is similar. To prove the bound on  $|T_3|$ , we double-count the number  $e(T, H)$  of edges in  $D$  from  $T$  to  $V(H)$ . Since  $d_C^-(x) \geq (n+k)/2 - |H|$  for any vertex  $x \in H$  we have that  $e(T, H) \geq |H|((n+k)/2 - |H|)$ . On the other hand  $e(T, H) \leq |T_3||H| + 2(|T| - |T_3|) = |T_3|(|H| - 2) + 2|T|$ . Before we can use this to estimate  $|T_3|$ , we need an upper bound on  $|T|$ . For this, recall that  $|F| \geq (n+k)/2 - |H|$ . Together with Corollary 7 this shows that  $|T| \leq |C| - |F| \leq (n-k)/2$ . Altogether this gives

$$\begin{aligned} |T_3| &\geq \frac{|H|((n+k)/2 - |H|) - (n-k)}{|H| - 2} = \frac{(|H| - 2)(n-k)/2 - |H|(|H| - k)}{|H| - 2} \\ &\geq \frac{n-k}{2} - |H| = \frac{|C| - |H| - k}{2}. \end{aligned}$$

The proof for  $|F_3|$  is similar. The bound on  $|T_3 \cup F_3|$  follows since  $|T_3 \cap F_3| \leq k$ . Indeed, the latter holds since Lemma 8 implies that whenever  $s, s' \in S$  are distinct and no vertex from  $S$  lies in the interior of  $sCs'$  then  $T_3 \cap F_3$  meets  $sCs'$  in at most one vertex.  $\square$

#### 4. PROOF OF THEOREM 1

Throughout this section, we assume that the order  $n$  of our given digraph  $D$  is sufficiently large compared to  $k$  for our estimates to hold. We will also omit floors and ceilings whenever this does not affect the argument. Let  $S$ ,  $C$  and  $H$  be as defined at the beginning of Section 3. Recall that we assume that  $C$  is not Hamiltonian and will show that we can extend  $C$  into a longer  $S$ -ordered cycle (which would yield a contradiction and thus would prove Theorem 1). Given consecutive vertices  $s, s' \in S$ , we call the path obtained from  $sCs'$  by deleting  $s'$  the *interval from  $s$  to  $s'$* . Thus no vertex from  $S$  lies in the interior of  $sCs'$  and  $C$  consists of precisely  $|S| = k$  disjoint intervals. In our proof of Theorem 1 we distinguish the following 4 cases according to the order of  $H$ . Recall that  $|H| \geq k$  by Lemma 6.

**Case 1.**  $k \leq |H| \leq 220k^3$ .

Recall that  $|T_3| \geq (n-k)/2 - |H| \geq n/3$  by Lemma 13 and so at least one of the  $k$  intervals of  $C$  must contain at least  $n/(3k)$  vertices from  $T_3$ . Suppose that this is the case for the interval  $I$  from  $s$  to  $s'$ . Recall that by Lemma 13 at most  $|H| + 2k \leq 3|H|$  vertices of  $C$  do not lie in  $T_3 \cup F_3$  and by Corollary 7 no vertex in  $F_3$  is the successor of a vertex in  $T_3$ . Since every maximal subpath of  $I$  consisting of vertices from  $T_3$  is succeeded by at least one vertex outside  $T_3 \cup F_3$ , it follows that  $I$  contains a subpath  $A$  which consists entirely of vertices from  $T_3$

and satisfies  $|A| \geq n/(3k(3|H| + 1))$ . Let  $A_1$  be the subpath of  $A$  consisting of its initial  $n/(20k|H|)$  inner vertices and let  $A_2$  be the subpath of  $A$  consisting of its last  $n/(20k|H|)$  inner vertices.

Let  $t$  be the first vertex of  $A$ . (So  $t^+$  is the first vertex of  $A_1$ .) Consider any vertex  $a$  on  $t^+Cs'$ . Lemma 8 implies that  $a \notin F$ . Thus  $N_D^-(a) \subseteq V(C)$  and hence

$$(2) \quad d_C^-(a) \geq \delta^0(D) \geq (n+k)/2 - 1 \geq n+k-1-|H|-|F| > |C|-|F|.$$

(To see the third inequality recall that  $|F| \geq (n+k)/2 - |H|$  by Lemma 13.)

**Case 1.1.** *There are vertices  $a_1 \in A_1$  and  $a_2 \in A_2$  such that  $a_1a_2$  is an edge.*

Inequality (2) applied with  $a := a_1^+$  implies that there exists a vertex  $w \in N_C^-(a_1^+)$  such that the successor  $w^+$  of  $w$  lies in  $F$ . Recall that  $F$  avoids  $t^+Cs'$  and so  $w^+$  must lie in  $s'Ct-s'$ . Hence  $w$  must lie in  $s'Ct-t$  (and thus in the interior of  $a_2Ca_1$ ). As  $a_2^- \in V(A) \subseteq T_3$  and as  $H$  is Hamiltonian connected by Lemma 6, there is an  $a_2^-w^+$  path  $P$  whose interior consists of precisely all the vertices in  $H$ . But then the  $S$ -ordered cycle  $a_1a_2Cwa_1^+Ca_2^-Pw^+Ca_1$  is Hamiltonian, contradicting the choice of  $C$  (see Figure 3).

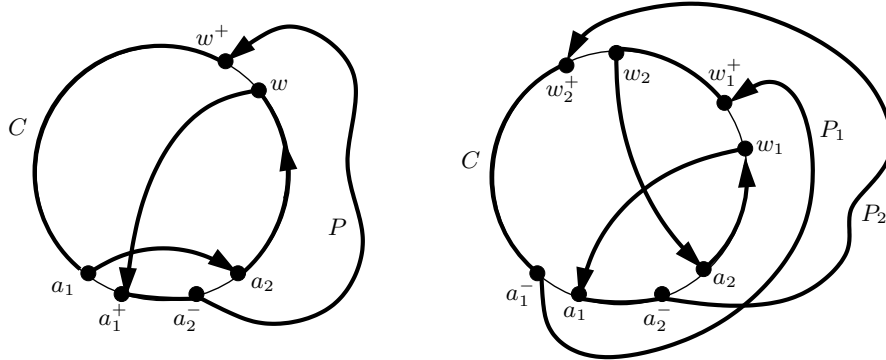


FIGURE 3. Extending  $C$  into a longer  $S$ -ordered cycle in Case 1.1 (left) and Case 1.2 (right).

**Case 1.2.** *There are no such vertices  $a_1 \in A_1$  and  $a_2 \in A_2$ .*

Let  $F_3^-$  denote the set of all predecessors of vertices in  $F_3$ . Recall that  $F$  avoids  $t^+Cs'$ . Thus  $F_3^-$  avoids  $tCs'-s'$ . Now consider any vertex  $a$  on  $A_2$ . Then  $N_D^-(a) \subseteq V(C)$  since  $a \notin F$  and thus  $N_D^-(a) \subseteq V(C) \setminus V(A_1)$  by our assumption. But then using that  $|F_3| \geq (n-k)/2 - |H|$  by Lemma 13 and arguing similarly as in (2) one can show that  $d_{C-A_1}^-(a) \geq n-1-|H|-|F_3| = |C|-1-|F_3| = |C-A_1|-|F_3^-|+|A_1|-1$ . Together with the fact that  $F_3^- \cap V(A_1) = \emptyset$  this gives

$$(3) \quad |N_{C-A_1}^-(a) \cap F_3^-| \geq |A_1| - 1 \geq n/(21k|H|).$$

Let  $I_1$  be the subpath of the interval  $I$  preceding the first vertex in  $A_1$ . So  $I_1 = sCt$ . Let  $I_2, \dots, I_k$  denote all the other intervals. For each  $i = 1, \dots, k$  let  $G_i$  be the auxiliary bipartite graph whose vertex classes are  $V(A_2)$  and  $V(I_i) \cap F_3^-$  and in which  $a \in V(A_2)$  is joined to  $w \in V(I_i) \cap F_3^-$  if  $wa \in E(D)$ . Recall that  $F_3^-$  avoids  $tCs'-s'$ . Thus  $F_3^- \subseteq V(I_1) \cup \dots \cup V(I_k)$  and so the edges of  $G_1 \cup \dots \cup G_k$

correspond to the edges of  $D$  from  $F_3^-$  to  $A_2$ . Together with (3) this implies that there is some  $i$  such that

$$e(G_i) \geq \frac{n|A_2|}{21k^2|H|} \geq \frac{n^2}{420k^3|H|^2} \geq 3n \geq 3|G_i|.$$

Thus  $G_i$  is not planar and so there are vertices  $a_1, a_2 \in V(A_2)$  and  $w_1, w_2 \in V(I_i) \cap F_3^-$  such that the edges  $w_1a_1, w_2a_2$  ‘cross’ in  $G_i$ , i.e. such that  $w_1$  lies in the interior of  $a_2Cw_2$  and  $a_1$  lies in the interior of  $w_2Ca_2$ . Recall that  $w_1^+, w_2^+ \in F_3$  by the definition of  $F_3^-$  and  $a_1^-, a_2^- \in T_3$  as  $A_2$  consisted of inner vertices of  $A$ . Thus we can apply Corollary 12 to obtain disjoint  $a_j^-w_j^+$  paths  $P_j$  having all their inner vertices in  $H$  and such that each  $P_j$  contains at least one inner vertex (where  $j = 1, 2$ ). Thus  $a_1^-P_1w_1^+Cw_2a_2Cw_1a_1Ca_2^-P_2w_2^+Ca_1^-$  is an  $S$ -ordered cycle with at least  $|C| + 2$  vertices (note that it contains all the vertices of  $C$ ), contradicting the choice of  $C$  (see Figure 3).

**Case 2.**  $220k^3 \leq |H| \leq n/2 - n/(50k)$ .

The argument for this case is similar to that in Case 1. Recall that  $|T_3| \geq (n - k)/2 - |H| \geq n/(60k)$  by Lemma 13 and so one of the  $k$  intervals of  $C$  must contain at least  $n/(60k^2)$  vertices from  $T_3$ . Suppose that this is the case for the interval  $I$  from  $s$  to  $s'$ . Let  $t$  be the first vertex on  $I$  that lies in  $T_3$ . Let  $A$  be the set consisting of the last  $n/(70k^2)$  vertices from  $T_3$  lying in the interior of  $I$ . For each  $a \in A$  let  $Q_a$  be the set of  $220k^3$  vertices of  $C$  preceding  $a$ . Note that the definition of  $A$  implies that  $Q_a$  lies in the interior of  $I$  and that  $t$  precedes the first vertex of  $Q_a$ . Together with Lemma 8 this shows that  $F$  avoids  $t^+Cs'$  and thus all of  $Q_a \cup \{a, a^+\}$ . In particular,  $a \notin F$ . Thus  $N_D^-(a) \subseteq V(C)$  and so  $a$  satisfies (2).

**Case 2.1.** *There is a vertex  $a \in A$  for which  $a^+$  receives an edge from some vertex  $q \in Q_a$ .*

Inequality (2) implies that there exists a vertex  $w \in N_C^-(a)$  such that the successor  $w^+$  of  $w$  lies in  $F$ . Note that  $w$  lies in the interior of  $aCq$  since  $F$  avoids  $Q_a \cup \{a, a^+\}$ . As  $a \in A \subseteq T_3$  and as  $H$  is Hamiltonian connected by Lemma 6, there is an  $a-w^+$  path  $P$  whose interior consists precisely of all the vertices in  $H$ . But then the cycle  $qa^+CwPw^+Cq$  is  $S$ -ordered and contains  $|H| - |Q_a| + 1 > 0$  more vertices than  $C$ , a contradiction.

**Case 2.2.** *There is no such vertex  $a \in A$ .*

This case is similar to Case 1.2. Let  $F_3^-$  denote the set of all predecessors of vertices in  $F_3$  again. Let  $A^+$  denote the set of all successors of vertices in  $A$ . Recall that  $F$  avoids  $t^+Cs'$ . Thus  $F_3^-$  avoids  $tCs' - s'$  and thus in particular all the sets  $Q_a$ .

Consider any  $a \in A$ . Then  $N_D^-(a^+) \subseteq V(C)$  since  $a^+ \notin F$  by Corollary 7. Thus  $N_D^-(a^+) \subseteq V(C) \setminus Q_a$  by our assumption. Hence similarly as in Case 1.2 one can show that  $d_{C-Q_a}^-(a^+) \geq |C - Q_a| - |F_3^-| + |Q_a| - 1$ . Together with the fact that  $F_3^- \cap Q_a = \emptyset$  this gives

$$(4) \quad |N_{C-Q_a}^-(a^+) \cap F_3^-| \geq |Q_a| - 1 \geq 210k^3.$$

Let  $I_1$  be the subpath of the interval  $I$  preceding the first vertex in  $A^+$ . Let  $I_2, \dots, I_k$  denote all the other intervals. For each  $i = 1, \dots, k$  let  $G_i$  be the auxiliary bipartite graph whose vertex classes are  $A^+$  and  $V(I_i) \cap F_3^-$  and in which  $a^+ \in A^+$  is

joined to  $w \in V(I_i) \cap F_3^-$  if  $wa^+$  is an edge of  $D$ . Note that  $F_3^- \subseteq V(I_1) \cup \dots \cup V(I_k)$  since  $F_3^-$  avoids  $tCs' - s'$ . Thus the edges of  $G_1 \cup \dots \cup G_k$  correspond to the edges from  $F_3^-$  to  $A^+$ . Together with (4) this implies that there is some  $i$  such that

$$e(G_i) \geq \frac{210k^3|A^+|}{k} = \frac{210k^3n}{70k^3} = 3n \geq 3|G_i|.$$

Thus  $G_i$  is not planar and so there are vertices  $a_1^+, a_2^+ \in V(A^+)$  and  $w_1, w_2 \in V(I_i) \cap F_3^-$  such that the edges  $w_1a_1^+, w_2a_2^+$  cross. As in Case 1.2 we can apply Corollary 12 to obtain disjoint  $a_j-w_j^+$  paths having all their inner vertices in  $H$  such that each  $P_j$  contains at least one inner vertex (where  $j = 1, 2$  and  $a_j$  is the predecessor of  $a_j^+$ ). Thus  $a_1P_1w_1^+Cw_2a_2^+Cw_1a_1^+Ca_2P_2w_2^+Ca_1$  is an  $S$ -ordered cycle with at least  $|C| + 2$  vertices (note that it contains all the vertices of  $C$ ), contradicting the choice of  $C$ .

**Case 3.**  $n/2 - n/(50k) \leq |H| \leq \lceil (n - k)/2 \rceil - 1$ .

Our first aim is to find vertices  $x_1, x_2, y_1, y_2$  on  $C$  with the following properties:

- (i)  $x_1, x_2, y_1, y_2$  occur on  $C$  in this order and either all of these vertices are distinct or else  $|\{x_1, x_2, y_1, y_2\}| = 3$  and  $x_1 = y_2$ .
- (ii)  $S$  avoids the interior of  $x_1Cx_2$ , the interior of  $y_1Cy_2$  as well as  $x_2$  and  $y_1$ .
- (iii) There are distinct vertices  $h_1, h_2, h'_1, h'_2 \in H$  such that  $x_1h_1, x_2h_2, h'_1y_1, h'_2y_2$  are edges.
- (iv) If  $x_1 \neq y_2$  (and so  $x_1, x_2, y_1, y_2$  are distinct) then there are disjoint  $x_i-y_i$  paths  $P_i$  of length at least 2 such that all inner vertices of  $P_1$  and  $P_2$  lie in  $H$  and  $|P_1 \cup P_2| \geq |H|/6$ .

To prove the existence of such vertices, suppose first that  $|T_8| \geq k + 1$  and  $|F_8| \geq k + 1$ . Then we can find two vertices  $x_1, x_2 \in T_8$  and two vertices  $y_1, y_2 \in F_8$  satisfying (ii). Then these vertices automatically satisfy (iii). Lemma 8 implies that they also satisfy (i). Finally, if they are all distinct then Corollary 12 shows that they also satisfy (iv).

So suppose next that for example  $|T_8| \leq k$  but  $|F_8| \geq k + 1$ . Pick  $y_1, y_2 \in F_8$  as before. To find  $x_1$  and  $x_2$ , first note that each vertex  $h \in H$  satisfies

$$d_C^-(h) \geq \delta^-(D) - (|H| - 1) \geq \lceil (n + k)/2 \rceil - 1 - \lceil (n - k)/2 \rceil + 2 = k + 1$$

and so  $h$  receives at least one edge from some vertex in  $T \setminus T_8$ . As each vertex in  $T \setminus T_8$  sends an edge to at most 7 vertices in  $H$ , this means that there are at least  $|H|/7$  independent edges from  $C$  to  $H$ . Thus the interior of some interval of  $C$  contains the endvertices of 16 of these independent edges which avoid  $y_1$  and  $y_2$ . Let  $X_1$  be the set of the first 8 endvertices of these edges on this interval and let  $X_2$  be the set of the next 8 endvertices. Then Lemma 11 implies that there are vertices  $x_1 \in X_1$  and  $x_2 \in X_2$  which together with  $y_1$  and  $y_2$  satisfy (iv). By construction,  $x_1, x_2, y_1, y_2$  are all distinct and satisfy (ii) and (iii). Again, Lemma 8 implies that they also satisfy (i). The cases when  $|T_8| \geq k + 1$  but  $|F_8| \leq k$  and when  $|T_8|, |F_8| \leq k$  are similar. So we have shown that there are vertices  $x_1, x_2, y_1, y_2$  satisfying (i)–(iv).

In what follows, we will frequently use the fact that any vertex  $x \in V(C) \setminus F_2$  receives an edge from all but at most

$$|C| - (\delta^-(D) - 1) \leq n/2 + n/(50k) - (n + k)/2 + 2 \leq n/(45k)$$

vertices of  $C$ . Similarly, any vertex  $x \in V(C) \setminus T_2$  sends an edge to all but at most  $n/(45k)$  vertices of  $C$ .

**Case 3.1.**  $|x_1 C x_2| \geq n/(15k)$

Let  $A_2$  be the set of  $n/(40k)$  vertices which immediately precede  $x_2$  and let  $A_1$  be the set of  $n/(40k)$  vertices which immediately precede  $A_2$ . Corollary 7 implies that the successor  $x_2^+$  of  $x_2$  on  $C$  does not lie in  $F$ . Thus  $x_2^+$  receives an edge from some vertex  $a_1 \in A_1$  since it receives an edge from all but at most  $n/(45k)$  vertices of  $C$ . Similarly, the predecessor  $y_2^-$  of  $y_2$  does not lie in  $T$  and thus sends an edge to some vertex  $a_2 \in A_2$ . Lemma 6 now implies that  $H$  contains a Hamilton path  $P$  from  $h_2$  to  $h'_2$ . But then the cycle  $a_1 x_2^+ C y_2^- a_2 C x_2 h_2 P h'_2 y_2 C a_1$  is  $S$ -ordered and contains all vertices of  $C$  except those in the interior of  $a_1 C a_2$  (see Figure 4). But as  $|H| > n/4 > |a_1 C a_2|$  this means that this new cycle is longer than  $C$ , a contradiction.

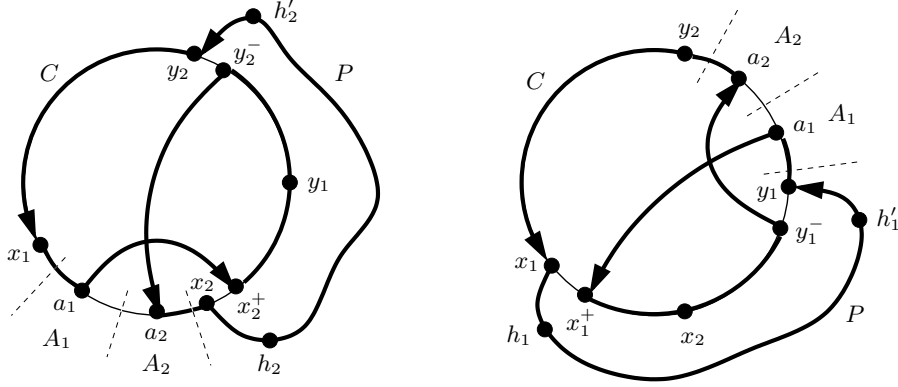


FIGURE 4. Extending  $C$  into a longer  $S$ -ordered cycle in Case 3.1 (left) and Case 3.2 (right).

**Case 3.2.**  $|y_1 C y_2| \geq n/(15k)$

The proof of this case is similar to that of Case 3.1. Let  $A_1$  be the set of  $n/(40k)$  vertices which immediately succeed  $y_1$  and let  $A_2$  be the set of  $n/(40k)$  vertices which immediately succeed  $A_1$ . Then the predecessor  $y_1^-$  of  $y_1$  sends an edge to some vertex  $a_2 \in A_2$  and the successor  $x_1^+$  of  $x_1$  receives an edge from some vertex  $a_1 \in A_1$ . Then the  $S$ -ordered cycle  $y_1^- a_2 C x_1 h_1 P h'_1 y_1 C a_1 x_1^+ C y_1^-$  is longer than  $C$ , where  $P$  is a Hamilton path in  $H$  from  $h_1$  to  $h'_1$  (see Figure 4).

**Case 3.3.**  $|y_2 C x_1| \geq n/5$

Let  $Z$  be a segment of the interior of  $y_2 C x_1$  such that  $|Z| \geq n/(6k)$  and such that  $Z$  avoids  $S$ . Let  $Z_1$  be the set consisting of the first  $n/(40k)$  vertices on  $Z$ . Let  $Z_2$  be the set consisting of the next  $n/(40k)$  vertices and define  $Z_3, \dots, Z_6$  similarly. As by Corollary 7 the predecessor  $y_1^-$  of  $y_1$  does not lie in  $T$  it must send an edge to some vertex  $z_4 \in Z_4$ . Similarly the predecessor  $y_2^-$  of  $y_2$  sends an edge to some vertex  $z_2 \in Z_2$ , the successor  $x_1^+$  of  $x_1$  receives an edge from some vertex  $z_5 \in Z_5$  and the successor  $x_2^+$  of  $x_2$  receives an edge from some vertex  $z_3 \in Z_3$ . Now Lemma 8 implies that either  $Z_1 \cap T_2 = \emptyset$  or  $Z_6 \cap F_2 = \emptyset$  or both. If  $Z_1 \cap T_2 = \emptyset$  then every

vertex in  $Z_1$  sends an edge to  $Z_6$  (since every vertex outside  $T_2$  sends an edge to all but at most  $n/(45k)$  vertices on  $C$ ). Similarly, if  $Z_6 \cap F_2 = \emptyset$  then every vertex in  $Z_6$  receives an edge from some vertex in  $Z_1$ . So in both cases we can find a  $Z_1$ - $Z_6$  edge  $z_1z_6$ . But then the cycle  $x_1P_1y_1Cy_2^-z_2Cz_3x_2^+Cy_1^-z_4Cz_5x_1^+Cx_2P_2y_2Cz_1z_6Cx_1$  is  $S$ -ordered and contains at least  $|P_1 \cup P_2| - 4 - (|Z| - 6) \geq |H|/6 - n/(6k) > 0$  vertices more than  $C$ , a contradiction (see Figure 5).

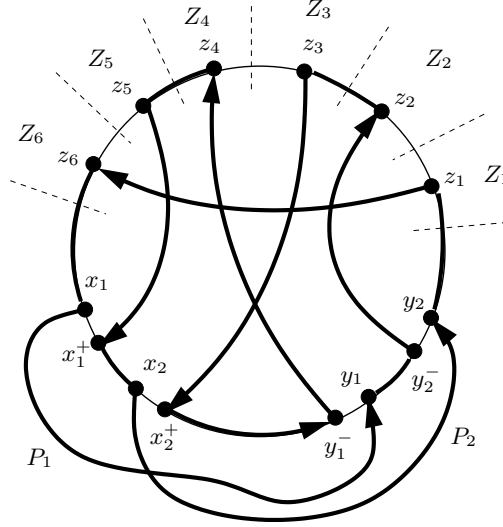


FIGURE 5. Extending  $C$  into a longer  $S$ -ordered cycle in Case 3.3.

**Case 3.4.** *None of Cases 3.1–3.3 holds.*

In this case we must have that  $|x_2Cy_1| \geq n/5$  and can argue similarly as in Case 3.3 (see Figure 6). We omit the details.

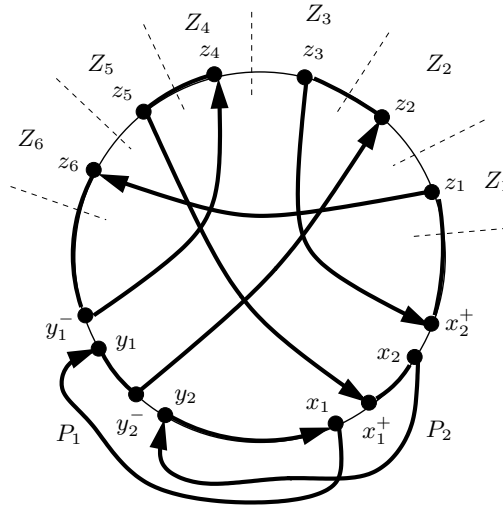


FIGURE 6. Extending  $C$  into a longer  $S$ -ordered cycle in Case 3.4.

**Case 4.** *None of Cases 1–3 holds.*

Together with Lemma 6 this implies that  $n - k$  is even and  $|H| = (n - k)/2$ . So  $|C| = (n + k)/2$ . First note that any vertex  $h \in H$  satisfies

$$(5) \quad d_C^+(h), d_C^-(h) \geq (n + k)/2 - 1 - (|H| - 1) = k.$$

Moreover, if  $h, h' \in H$  are distinct and if  $s \in S \cap N_C^-(h)$  then by Lemma 8 the special vertex  $s'$  succeeding  $s$  on  $C$  (i.e. the unique vertex  $s' \in S$  for which  $S$  avoids the interior of  $sCs'$ ) cannot lie in  $N_C^+(h')$ . Thus  $|S \cap N_C^-(h)| + |S \cap N_C^+(h')| \leq k$  and so

$$(6) \quad |N_C^-(h) \setminus S| + |N_C^+(h') \setminus S| \geq |N_C^-(h)| + |N_C^+(h')| - k \stackrel{(5)}{\geq} k.$$

**Case 4.1.** *There exists some vertex  $x \in N_C^-(h) \setminus S$ .*

First note that by Corollary 7 the successor  $x^+$  of  $x$  on  $C$  does not lie in  $F$ . Thus  $d_C^-(x^+) \geq \delta^0(D) = |C| - 1$  and so  $x^+$  receives an edge from the predecessor  $x^-$  of  $x$ . Pick any vertex  $y \in F \setminus \{x, x^-\}$ . (Such a vertex exists since  $|F| \geq 3$  by (5).) Note that  $y \neq x^+$  since  $x^+ \notin F$ . By Corollary 7 the predecessor  $y^-$  of  $y$  does not send an edge to  $H$  and so  $y^-x$  must be an edge (since  $d_C^+(y^-) = |C| - 1$ ). Now apply Lemma 6 to find an  $x$ - $y$  path  $P$  of length at least 2 all whose inner vertices lie in  $H$ . Then  $x^-x^+Cy^-xPyCx^-$  is an  $S$ -ordered cycle which is longer than  $C$ , a contradiction.

**Case 4.2.** *There is no vertex as in Case 4.1.*

Together with (6) this implies that we can find a vertex  $x \in N_C^+(h') \setminus S$ . We then argue similarly as in Case 4.1. This completes the proof of Theorem 1.

## 5. ACKNOWLEDGEMENT

We are grateful to Oliver Cooley for a careful reading of the manuscript.

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Daniela Kühn, Deryk Osthus & Andrew Young

School of Mathematics

University of Birmingham

Edgbaston

Birmingham

B15 2TT

UK

*E-mail addresses:* `{kuehn,osthus,younga}@maths.bham.ac.uk`